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A proximal decomposition algorithm for variational inequality problems

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Abstract

In this paper, we propose a new decomposition algorithm for solving monotone variational inequality problems with linear constraints. The algorithm utilizes the problem's structure conductive to decomposition. At each iteration, the algorithm solves a system of nonlinear equations, which is structurally much easier to solve than variational inequality problems, the subproblems of classical decomposition methods, and then performs a projection step to update the multipliers. We allow to solve the subproblems approximately and we prove that under mild assumptions on the problem's data, the algorithm is globally convergent. We also report some preliminary computational results, which show that the algorithm is encouraging.

Keywords: Variational inequality problems; Decomposition algorithms; Global convergence; Monotone mappings

1. Introduction

Let $S \subset \mathbb{R}^n$ be a nonempty closed convex subset of $\mathbb{R}^n$ and let f be a mapping from $\mathbb{R}^n$ into itself. A classical variational inequality problem, denoted by VI(f,S), is to find a vector $x^* \in S$ such that

$$f(x^*)^\top (z - x^*) \geqslant 0, \quad \forall z \in S. \tag{1}$$

In many problems arising from traffic equilibrium and network economic problems [1,2,21], S often has the following structure:

$$S = S_1 = \{x \in \mathbb{R}^n \mid Bx = b, x \geqslant 0\} \tag{2}$$

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or

$$S = S_2 = \{x \in \mathbb{R}^n \mid Bx \geqslant b, x \geqslant 0\} \tag{3}$$

where $B \in R^{m\times n}$ is a given matrix and $b \in \mathbb{R}^m$ is a given vector.

Typically (see, for example, [6–9,19,12,13,10]), for solving the primal problem (1) with structure (2) or (3), we first introduce a Lagrange multiplier to the linear constraint $Bx = b$ ($Bx \geqslant b$) to transform the problem to an equivalent form and then solve the consequent variational inequality problem, denoted by VI(Q, W), of finding $w^* \in W$, such that

$$Q(w^*)^\top (w - w^*) \geqslant 0 \quad \forall w \in W,$$

where

$$w = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Q(w) = \begin{pmatrix} f(x) - B^\top y \\ Bx - b \end{pmatrix}, \quad W = \mathbb{R}^n_+ \times Y,$$

and $Y = \mathbb{R}^m$ if $S = S_1$ or $Y = \mathbb{R}^m_+$ if $S = S_2$.

For solving the variational inequality problem (1) with $S = S_1$, Gabay [7] and Gabay and Mercier [8] proposed the following decomposition method, which is called *alternating direction method*:

Given $(x^k, y^k) \in \mathbb{R}^n_+ \times \mathbb{R}^m$, find $x^{k+1} \geqslant 0$, such that

$$(x' - x^{k+1})^\top \{f(x^{k+1}) - B^\top [y^k - (Bx^{k+1} - b)]\} \geqslant 0 \quad \forall x' \geqslant 0, \tag{4}$$

then update y via

$$y^{k+1} = y^k - (Bx^{k+1} - b).$$

Note that this algorithm can also be used to solve the variational inequality problem with $S = S_2$ by introducing a slack vector to the linear inequality constraint to transform S_2 to the same form as S_1,

$$S_2 = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid Bx - z = b, \ z \geqslant 0\}.$$

However, this will increase the dimension of subproblem (4) from n to $n + m$.

Then, for solving the variational inequality problem (1) with $S = S_2$, another decomposition method was proposed [6,9], which is called *method of multiplier*

Given $(x^k, z^k, y^k) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m$, find $x^{k+1} \geqslant 0$, such that

$$(x' - x^{k+1})^\top \{f(x^{k+1}) - B^\top [y^k - (Bx^{k+1} - z^k - b)]\} \geqslant 0 \quad \forall x' \geqslant 0, \tag{5}$$

then update z via

$$z^{k+1} = \max\{0, Bx^{k+1} - y^k - b\}$$

and update y via

$$y^{k+1} = y^k - (Bx^{k+1} - z^k - b).$$

These decomposition methods are attractive for large-scale problems, since they decompose the original problems into a series of subproblems with lower scale. However, note that both (4) and (5) are still variational inequality problems, which are structurally difficult to solve.

To overcome this difficulty, recently, He and Zhou [19] proposed a new decomposition method for solving convex quadratic programming problems. Their method is more attractive than those in [7,8] since at each iteration, instead of solving the variational inequality problem (4), it only makes some matrix–vector products. Their algorithm was then extended to linear variational inequality problems [12] and to nonlinear variational inequalities with co-coercive mappings [13]. Most recently, by adopting a self-adaptive Armijo-type line search strategy, the method was extended to nonlinear variational inequalities with monotone mappings [10].

Inspired by these, in this paper, we propose a new decomposition algorithm for solving variational inequality problems with $S = S_1$ or $S = S_2$. At each iteration, instead of solving the structurally difficult problems (4) or (5), we solve a system of well-conditioned nonlinear equations with respect to the variable x. Then, we perform a projection step to generate the next iterate. We allow to solve the equations approximately, which makes the algorithm more practical. The accuracy criterion we adopt here is the one developed recently by Solodov and Svaiter [23], which is more constructive than the classical one assuming the summable or the square summable of the sequence of the error tolerance parameters [22,11]. We prove that under mild assumptions that the underlying mapping f is continuous and monotone and the solutions set is nonempty, the sequence generated by the algorithm converges to a solution globally.

The remainder of the paper is organized as follows. In Section 2, we summarize some basic definitions and properties to be used in this paper. In Section 3, the new decomposition algorithm is described formally and its global convergence is proved in Section 4 under mild condition that the underlying mapping f is continuous and monotone. In Section 5, we report some preliminary computational results of the proposed method, and Section 6 gives some concluding remarks.

2. Preliminaries

In this section, we summarize some basic concepts and their properties that will be useful in the subsequent sections.

First, we denote $\|x\| = \sqrt{x^\top x}$ as the Euclidean norm. Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and let $P_K[\,\cdot\,]$ denote the projection mapping from $\mathbb{R}^n$ onto K. The following well-known properties of the projection operator will be used below.

Lemma 2.1. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ and any $z \in K$, the following properties hold*:

1. $(x - P_K[x])^\top (z - P_K[x])\rangle \leqslant 0$.
2. $\|P_K[x] - P_K[y]\|^2 \leqslant \|x - y\|^2 - \|P_K[x] - x + y - P_K[y]\|^2$.

It is well known that the VI(f, K) is equivalent to the projection equation

$$x = P_K[x - \beta f(x)],$$

where β is an arbitrary positive constant. Let

$$e(x, \beta) = x - P_K[x - \beta f(x)]$$

denote the residual function of the projection equation, then $\mathrm{VI}(f,K)$ is equivalent to finding a zero point of $e(x,\beta)$. That is,

$$u \text{ is a solution of the problem} \Leftrightarrow e(u,\beta)=0.$$

In the literatures [15–17], $\|e(u,\beta)\|$ was viewed as a measure function, which measures how much u fails to be a solution point.

We need the following definitions concerning the functions.

Definition 2.2. (a) A mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone, if

$$(x-y)^\top(f(x)-f(y)) \geqslant 0 \quad \forall x,y\in\mathbb{R}^n.$$

(b) A mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ is said to be strongly monotone with modulus $\gamma>0$, if

$$(x-y)^\top(f(x)-f(y)) \geqslant \gamma\|x-y\|^2 \quad \forall x,y\in\mathbb{R}^n.$$

In the following, we always suppose that the underlying mapping f of the variational inequality problem under consideration is continuous and monotone.

3. The decomposition algorithm

Note that, by introducing a Lagrange multiplier y to the linear constraint and the nonnegative constraint, we can transform $\mathrm{VI}(f,S_1)$ and $\mathrm{VI}(f,S_2)$ to the uniform description of finding a vector $u^*\in\Omega$, such that

$$F(u^*)^\top(u-u^*) \geqslant 0 \quad \forall u\in\Omega, \tag{6}$$

where

$$u=\begin{pmatrix} x \\ y \end{pmatrix}, \quad F(u)=F(x,y)=\begin{pmatrix} f(x)-A^\top y \\ Ax-a \end{pmatrix}, \quad \Omega=\mathbb{R}^n\times Y \tag{7}$$

and

$$A=\begin{pmatrix} B \\ I \end{pmatrix}, \quad a=\begin{pmatrix} b \\ 0 \end{pmatrix},$$

Y is a set in R^{m+n} with $Y=Y_{\mathrm{I}}\times Y_{II}$, $Y_{\mathrm{I}}\subset\mathbb{R}^m$ and $Y_{\mathrm{II}}=\mathbb{R}^n_+$. The only difference is $Y_{\mathrm{I}}=\mathbb{R}^m$ if $S=S_1$ and $Y_{\mathrm{I}}=\mathbb{R}^m_+$ if $S=S_2$. In this paper, we focus our attention on the structured variational inequality problem (6) and (7).

We are now in a position to describe our method formally.

Algorithm 3.1. An inexact decomposition algorithm.

Step 0: Choose an arbitrary initial point $u^0=(x^0,y^0)\in\mathbb{R}^n\times Y$, and parameters $\varepsilon>0$, $\sigma\in(0,1)$, $\underline{c}\in(0,(1-\sigma)/(4\|A\|^2))$. Set $k:=0$.

Step 1: Choose $c_k \in [\underline{c}, (1-\sigma)/\|A\|^2]$ and find $\bar{x}^k \in \mathbb{R}^n$ by solving the following system of nonlinear equations

$$c_k(f(\cdot) - A^\top y^k) + (\cdot - x^k) = r^k(\bar{x}^k), \tag{8}$$

such that

$$\|r^k(\bar{x}^k)\| \leqslant \sigma \|x^k - \bar{x}^k\|. \tag{9}$$

Step 2: Set

$$\bar{y}^{\,k} = P_Y[y^k - (A\bar{x}^k - a)] \tag{10}$$

and

$$g(u^k) = g(x^k, y^k) = \begin{pmatrix} f(\bar{x}^k) - A^\top \bar{y}^{\,k} \\ y^k - \bar{y}^{\,k} \end{pmatrix}.$$

Then compute α_k by

$$\alpha_k = g(u^k)^\top (u^k - \bar{u}^{\,k}) / \|g(u^k)\|^2. \tag{11}$$

Step 3: Compute $u^{k+1} = (x^{k+1}, y^{k+1})$ via

$$u^{k+1} = u^k - \alpha_k g(u^k). \tag{12}$$

Step 4: If

$$\|x^k - \bar{x}^k\| + \|y^k - \bar{y}^{\,k}\| \leqslant \varepsilon$$

then stop. Otherwise, Set $k := k + 1$ and goto Step 1.

In Step 1, to find the solution $\bar{x}^k$, one can solve the equation

$$c_k(f(\cdot) - A^\top y^k) + (\cdot - x^k) = 0 \tag{13}$$

by Newton's method [5,3] (with starting point $x_0 := x^k$) and stop with the first Newton iterate satisfying (9). Note that this equation is of the form

$$x + c_k f(x) = d^k.$$

Since f is continuous and monotone, the mapping $I + c_k f$ is strongly monotone. This system of nonlinear equations is well conditioned. Moreover, since for k large enough, x^k is close to $\bar{x}^k$ (see the following proof), Newton's method can find a solution for this equation within finitely many iterations. Note also that the equations is much easier to solve than variational inequality problems (4) and (5). Furthermore, we allow to solve it approximately with accuracy criterion (9), which is more constructive than the classical one assuming summable or square summable of the sequence of error tolerance, see [22,11,18].

If $x^k = \bar{x}^k$ and $y^k = \bar{y}^{\,k}$, then it follows from (9) that $r^k(\bar{x}^k) = 0$. Hence, from (8) and (10), we have

$$f(x^k) - A^\top y^k = 0$$

and

$$y^k = P_Y[y^k - (Ax^k - a)],$$

which mean that (x^k, y^k) is a solution of $\mathrm{VI}(F, \Omega)$. On the other hand, if (x^k, y^k) is a solution of $\mathrm{VI}(F, \Omega)$, then we have $x^k = \bar{x}^k$, $y^k = \bar{y}^k$. That is

$$(x^k, y^k) \text{ is a solution of } \mathrm{VI}(F, \Omega) \iff \|x^k - \bar{x}^k\| = \|y^k - \bar{y}^k\| = 0.$$

We thus can use $\|x^k - \bar{x}^k\| + \|y^k - \bar{y}^k\|$ as a measure function, which measures how much that (x^k, y^k) fails to be a solution of $\mathrm{VI}(F, \Omega)$. The stop criterion in Step 4 is reasonable.

4. Global convergence

In this section, we analyze the global convergence of the proposed algorithm under mild conditions that the underlying mapping f is continuous and monotone and the solution set of $\mathrm{VI}(F, \Omega)$ (6) and (7), denoted by Ω^*, is nonempty.

First, note that Y is a nonempty closed convex subset of $\mathbb{R}^{m+n}$. Let $u^* = (x^*, y^*) \in \Omega^*$ is an arbitrary solution of $\mathrm{VI}(F, \Omega)$. Then, it follows from Lemma 2.1 that

$$\{y^k - (A\bar{x}^k - a) - P_Y[y^k - (A\bar{x}^k - a)]\}^\top \{P_Y[y^k - (A\bar{x}^k - a)] - y^*\} \geqslant 0.$$

Since u^* is a solution of $\mathrm{VI}(F, \Omega)$ and $P_Y[\,\cdot\,] \in Y$, it follows from (6) that

$$(Ax^* - a)^\top (P_Y[y^k - (A\bar{x}^k - a)] - y^*) \geqslant 0.$$

Adding the above two inequalities, we have

$$\{y^k - \bar{y}^k - A(\bar{x}^k - x^*)\}^\top \{(y^k - y^*) - (y^k - \bar{y}^k)\} \geqslant 0,$$

which is equivalent to the inequality

$$\begin{aligned}
&(x^k - x^*)^\top (A^\top (y^k - \bar{y}^k)) + (y^k - y^*)^\top (y^k - \bar{y}^k) \\
&\quad \geqslant (y^k - y^*)^\top (A\bar{x}^k - Ax^*) + \|y^k - \bar{y}^k\|^2 - (A\bar{x}^k - Ax^k)^\top (y^k - \bar{y}^k).
\end{aligned} \tag{14}$$

Since u^* is a solution of $\mathrm{VI}(F, \Omega)$, we have that

$$f(x^*) = A^\top y^*.$$

From the monotonicity of f, we have

$$\begin{aligned}
(f(\bar{x}^k) - A^\top y^k)^\top (\bar{x}^k - x^*) &= ((f(\bar{x}^k) - f(x^*)) - A^\top (y^k - y^*))^\top (\bar{x}^k - x^*) \\
&\geqslant -(y^k - y^*))^\top (A\bar{x}^k - Ax^*).
\end{aligned} \tag{15}$$

Since

$$f(\bar{x}^k) - A^\top y^k = \frac{1}{c_k}[x^k - \bar{x}^k + r^k(\bar{x}^k)],$$

we have

$$(f(\bar{x}^k) - A^\top y^k)^\top (x^k - \bar{x}^k) = \frac{1}{c_k}(x^k - \bar{x}^k + r^k(\bar{x}^k))^\top (x^k - \bar{x}^k)$$
$$\geqslant \frac{1}{c_k}(\|x^k - \bar{x}^k\|^2 - \|r^k(\bar{x}^k)\|\|x^k - \bar{x}^k\|)$$
$$\geqslant \frac{1-\sigma}{c_k}\|x^k - \bar{x}^k\|^2, \tag{16}$$

where the first inequality follows from the Cauchy–Schwarz inequality and the last one follows from (9). It follows from (15) and (16) that

$$(f(\bar{x}^k) - A^\top y^k)^\top (x^k - x^*)$$
$$= (f(\bar{x}^k) - A^\top y^k)^\top (\bar{x}^k - x^*) + (f(\bar{x}^k) - A^\top y^k)^\top (x^k - \bar{x}^k)$$
$$\geqslant -(y^k - y^*))^\top (A\bar{x}^k - Ax^*) + \frac{1-\sigma}{c_k}\|x^k - \bar{x}^k\|^2. \tag{17}$$

Adding (14) and (17), we have

$$(x^k - x^*)^\top (f(\bar{x}^k) - A^\top \bar{y}^k) + (y^k - y^*)^\top (y^k - \bar{y}^k)$$
$$\geqslant g(u^k)^\top (u^k - \bar{u}^k)$$
$$= (f(\bar{x}^k) - A^\top y^k)^\top (x^k - \bar{x}^k) + \|y^k - \bar{y}^k\|^2 + (Ax^k - A\bar{x}^k)^\top (y^k - \bar{y}^k) \tag{18}$$

$$\geqslant \frac{1-\sigma}{c_k}\|x^k - \bar{x}^k\|^2 + \frac{1}{2}\|y^k - \bar{y}^k\|^2 - \frac{\|A\|^2}{2}\|x^k - \bar{x}^k\|^2$$
$$\geqslant \frac{\|A\|^2}{2}\|x^k - \bar{x}^k\|^2 + \frac{1}{2}\|y^k - \bar{y}^k\|^2, \tag{19}$$

where the last inequality follows from the choice of c_k.

In fact, the above analysis has proved the following lemma.

Lemma 4.1. *If* $u^k = (x^k, y^k)$ *is not a solution of* $\mathrm{VI}(F, \Omega)$, *then* $-g(u^k)$ *is a descent direction of the merit function* $\frac{1}{2}\|u - u^*\|^2$, *though* u^* *is unknown.*

The following lemma is essential to establish the global convergence of the proposed algorithm.

Lemma 4.2. *Suppose that* f *is continuous and monotone, the solution set* Ω^* *of* $\mathrm{VI}(F, \Omega)$ *is nonempty. Then*

1. *The generated sequence* $\{u^k\} = \{(x^k, y^k)\}$ *is bounded.*
2. *The sequence* $\{\bar{u}^k\} = \{(\bar{x}^k, \bar{y}^k)\}$ *is bounded.*
3. $\lim_{k\to\infty} \|x^k - \bar{x}^k\| = \lim_{k\to\infty} \|y^k - \bar{y}^k\| = 0.$

Proof. It follows from (19) that

$$
\begin{aligned}
& g(u^k)^\top (u^k - \bar{u}^k) \\
& \quad = (f(\bar{x}^k) - A^\top y^k)^\top (x^k - \bar{x}^k) + \|y^k - \bar{y}^k\|^2 - (A\bar{x}^k - Ax^k)^\top (y^k - \bar{y}^k) \\
& \quad \geqslant \frac{\|A\|^2}{2} \|x^k - \bar{x}^k\|^2 + \frac{1}{2} \|y^k - \bar{y}^k\|^2.
\end{aligned}
$$

Using again the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\|g(u^k)\|^2 &= \|f(\bar{x}^k) - A^\top \bar{y}^k\|^2 + \|y^k - \bar{y}^k\|^2 \\
&\leqslant 2(\|f(\bar{x}^k) - A^\top \bar{y}^k\|^2 + \|A\|^2 \|y^k - \bar{y}^k\|^2) + \|y^k - \bar{y}^k\|^2 \\
&= 2 \left\| \frac{1}{c_k}(x^k - \bar{x}^k + r^k(\bar{x}^k)) \right\|^2 + (1 + 2\|A\|^2) \|y^k - \bar{y}^k\|^2 \\
&\leqslant 4 \frac{(1+\sigma^2)}{c_k^2} \|x^k - \bar{x}^k\|^2 + (1 + 2\|A\|^2) \|y^k - \bar{y}^k\|^2 \\
&\leqslant 4 \frac{(1+\sigma^2)}{\underline{c}^2} \|x^k - \bar{x}^k\|^2 + (1 + 2\|A\|^2) \|y^k - \bar{y}^k\|^2.
\end{aligned}
$$

Thus, there exists a constant $\tau > 0$, such that for all $k \geqslant 0$,

$$
\begin{aligned}
\alpha_k &= g(u^k)^\top (u^k - \bar{u}^k) / \|g(u^k)\|^2 \\
&\geqslant \tau. \qquad (20)
\end{aligned}
$$

From (12), we have

$$
\begin{aligned}
& \|x^{k+1} - x^*\|^2 + \|y^{k+1} - y^*\|^2 \\
& \quad = \|x^k - x^*\|^2 + \|y^k - y^*\|^2 + \alpha_k^2 \|g(u^k)\|^2 \\
& \qquad - 2\alpha_k (f(\bar{x}^k) - A^\top \bar{y}^k)^\top (x^k - x^*) - 2\alpha_k (y^k - \bar{y}^k)^\top (y^k - y^*) \\
& \quad \leqslant \|x^k - x^*\|^2 + \|y^k - y^*\|^2 \\
& \qquad - \alpha_k \{ (f(\bar{x}^k) - A^\top y^k)^\top (x^k - \bar{x}^k) + \|y^k - \bar{y}^k\|^2 - (A\bar{x}^k - Ax^k)^\top (y^k - \bar{y}^k) \} \\
& \quad \leqslant \|x^k - x^*\|^2 + \|y^k - y^*\|^2 - \tau \left(\frac{\|A\|^2}{2} \|x^k - \bar{x}^k\|^2 + \frac{1}{2} \|y^k - \bar{y}^k\|^2 \right), \qquad (21)
\end{aligned}
$$

where the first inequality follows from (11) and (18) and the last one follows from (20) and (19).

It follows from (21) that

$$
\|x^{k+1} - x^*\|^2 + \|y^{k+1} - y^*\|^2 \leqslant \cdots \leqslant \|x^0 - x^*\|^2 + \|y^0 - y^*\|^2.
$$

The generated sequence $\{u^k = (x^k, y^k)\}$ is thus bounded.

Also from (21), it follows that

$$\sum_{k=0}^{\infty}\|x^k-\bar{x}^k\|^2<\infty,$$

$$\sum_{k=0}^{\infty}\|y^k-\bar{y}^k\|^2<\infty.$$

We then get assertions 2 and 3 immediately. □

We are now ready to prove the main theorem in this section.

Theorem 4.3. *Let the assumptions in Lemma* 4.2 *hold. Then the whole sequence* $\{u^k\}$ *generated by Algorithm* 3.1 *converges to a solution of* $\mathrm{VI}(F,\Omega)$ *globally.*

Proof. It follows from Lemma 4.2 that $\{u^k\}$ is bounded. It thus has at least one cluster point. Let $\tilde{u}\in\Omega$ be a cluster point of $\{u^k\}$ and let $\{u^{k_j}\}$ be the corresponding subsequence converging to $\tilde{u}$. It follows from Lemma 4.2 that $\bar{u}^{k_j}\to\tilde{u}$. From (9) and Lemma 4.2, we have that

$$\lim_{k\to\infty}\|r^k(\bar{x}^k)\|=0.$$

Taking limit in (8) and (10) along the subsequence and using the continuity of f and the projection operator $P_Y[\cdot]$, we have

$$f(\tilde{x})-A^{\top}\tilde{y}=0$$

and

$$\tilde{y}=P_Y[\tilde{y}-(A\tilde{x}-a)],$$

which mean that $\tilde{u}=(\tilde{x},\tilde{y})\in\Omega$ is a solution of $\mathrm{VI}(F,\Omega)$. Since $u^*=(x^*,y^*)\in\Omega^*$ is an arbitrary solution of $\mathrm{VI}(F,\Omega)$, we can just take $u^*=\tilde{u}$ in the above analysis and thus

$$\|x^{k+1}-\tilde{x}\|^2+\|y^{k+1}-\tilde{y}\|^2\leqslant\|x^k-\tilde{x}\|^2+\|y^k-\tilde{y}\|^2.$$

The whole sequence $\{u^k=(x^k,y^k)\}$ thus converges to $\tilde{u}$, a solution of $\mathrm{VI}(F,\Omega)$. □

5. Numerical results

To test the ability of the proposed algorithm, in this section, we implement it in MATLAB. The examples used here are taken from the test problems of Taji et al. [24], which are modifications of the test problems of Marcotte and Dussault [20]. The constraint set S and the mapping f are taken, respectively, as

$$S=S_2=\left\{x\in R^5\;\middle|\;\sum_{i=1}^{5}x_i\geqslant 10,\ x_i\geqslant 0,\ i=1,2,\dots,5\right\}$$

and

$$f(x) = Mx + \rho C(x) + q,$$

where M is a 5×5 asymmetric positive definite matrix and $C_i(x) = \arctan(x_i - 2)$, $i = 1, 2, \ldots, 5$. The parameter ρ is used to vary the degree of asymmetry and nonlinearity. The data of this example are given as follows:

$$f(x) = \begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.934 & 1.007 \\ 1.063 & 0.567 & -1.144 & 0.550 & -0.548 \\ -0.259 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$+ \rho \begin{pmatrix} \arctan(x_1 - 2) \\ \arctan(x_2 - 2) \\ \arctan(x_3 - 2) \\ \arctan(x_4 - 2) \\ \arctan(x_5 - 2) \end{pmatrix} + \begin{pmatrix} 5.308 \\ 0.008 \\ -0.938 \\ 1.024 \\ -1.312 \end{pmatrix}.$$

Thus,

$$B = (1, 1, 1, 1, 1) \quad \text{and} \quad b = 10.$$

In our formulation (6) and (7),

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Y = \mathbb{R}^6_+.$$

The problem has a unique solution $x^* = (2, 2, 2, 2, 2)^\top$ and the Lagrange multiplier is $y^* = (2, 0, 0, 0, 0, 0)^\top$.

For solving subproblem (8), we use the Newton method [3] to get an approximate solution satisfying the accuracy criterion (9). At each Newton step, we take the solution of the last main iteration as the starting point. The parameters used in the algorithm are set as $\sigma = 0.9$ and $c_k \equiv 0.1$ for all k. The stop parameter is set to be $\varepsilon = 10^{-6}$. Tables 1 and 2 report the computational results for $\rho = 10$ and 20, respectively. In these tables, n_k denotes the total Newton steps used to solve the subproblems.

Table 1
Numerical results for $\rho = 10$

Starting point	Number of Iter.	CPU (s)	$\|x^k - x^*\|$	n_k
$(25, 0, 0, 0, 0)$	14	0.06	2.40×10^{-7}	37
$(10, 0, 10, 0, 10)$	17	0.11	1.54×10^{-7}	42
$(10, 0, 0, 0, 0)$	12	0.06	5.87×10^{-7}	29
$(0, 2.5, 2.5, 2.5, 2.5)$	11	0.06	7.38×10^{-7}	23
$(0, 0, 0, 0, 0)$	8	0.06	8.85×10^{-7}	19
(1,1,1,1,1)	10	0.06	7.12×10^{-7}	22

Table 2
Numerical results for $\rho = 20$

Starting point	Number of Iter.	CPU (s)	$\|x^k - x^*\|$	n_k
$(25, 0, 0, 0, 0)$	17	0.16	4.77×10^{-7}	47
$(10, 0, 10, 0, 10)$	22	0.22	8.32×10^{-7}	49
$(10, 0, 0, 0, 0)$	14	0.06	3.57×10^{-7}	32
$(0, 2.5, 2.5, 2.5, 2.5)$	10	0.06	9.11×10^{-7}	21
$(0, 0, 0, 0, 0)$	12	0.06	1.51×10^{-7}	33
$(1, 1, 1, 1, 1)$	11	0.06	6.63×10^{-7}	22

The results in Tables 1 and 2 indicate that the new decomposition algorithm is quite efficient. At each main iteration, it needs about 2–3 Newton steps to get an approximate solution of subproblem (8) satisfying (9). Especially, when k is large enough, x^k is a good initial point of the subproblem. Though the iterative number is larger than Newton-type method [24], the total CPU time is smaller. Especially, the computational cost at each iteration is much smaller, since, at each iteration, the Newton-type method [24] needs to make some projections to the feasible set S, which is more difficult than to make projections to the nonnegative orthant of $\mathbb{R}^{n+m}$ and, it needs to solve a linear variational inequality problem at each iteration, which is also time consuming from the computational point of view.

The same problem with $\rho = 10$ was also considered in [25]. At each iteration, their algorithm also solves a system nonlinear equations with the same structure as (8). However, this subproblem has to be solved exactly. Additional to this, it has also to solve a linear variational inequality problem (LCP)

$$(y' - y^k)^\top \{(Ax^k - a) - A(f(x^k) - A^\top y^k)\} \geqslant 0 \quad \forall y' \geqslant 0,$$

to get y^k. Though this problem can be solved by the standard Lemke Algorithm [4], which can find a solution of LCP via finite steps, it is still time consuming, see [25, Table 1].

To show the advantage of this decomposition method for large-scale problems, we implement it to a set of spatial price equilibrium problems. The details of these problems follow from [19,13,14],

Table 3
Number of iterations for different scale and precisions

m	n	mn	$\varepsilon = 0.1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
5	5	25	4	8	44	56
5	10	50	6	24	35	61
5	20	100	10	22	58	96
10	10	100	12	30	50	85
10	20	200	15	33	156	173
20	30	600	24	53	109	175
30	40	1200	27	49	110	197
40	50	2000	30	72	131	796
50	60	3000	34	63	351	669

as follows:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}x_{ij} + \frac{1}{2}h_{ij}x_{ij}^2\right), \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1,\ldots,m, \\ & \sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1,\ldots,n, \\ & x_{ij} \geqslant 0, \end{aligned}$$

where s_i is the supply amount on the ith supply market, $i = 1,\ldots,m$ and d_j the demand amount on the jth demand market, $j = 1,\ldots,n$.

We use the same cost function as in [19]:

$$c_{ij} \in (0,100) \quad \text{and} \quad h_{ij} \in (0.005, 0.01).$$

The parameters s_i and d_j are generated randomly in $(0,100)$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. For this problem, the associate mapping of the variational inequality problem is linear. Thus, the computation burden per iteration is just function evaluation. The calculations were started with u^0 generated randomly in $(0,100)$ and stopped for some prescribed $\varepsilon > 0$. The computational results are summarized in Table 3 for some m and n.

The results in Table 3 show that the required iterative numbers are relatively small as compared with the size of problems. As this decomposition method only requires function evaluations per iteration, it is attractive from a computational point of view.

6. Concluding remarks

In this paper, we proposed a new decomposition algorithm for solving variational inequality problems with linear equality constraints or inequality constraints in a uniform framework. At each iteration, the algorithm solves a system of well-conditioned nonlinear equations with respect to x, the primal variable, and then performs a projection step to generated the next iterate. Furthermore, we allow to solve the subproblem approximately with a constructive accuracy criterion. The algorithm is thus well comparable to the original decomposition algorithms, which solves a series of variational inequality problems, a class of problems that is structurally much more difficult to solve than system of equations. The proposed algorithm is also well comparable to [25], which solves $\mathrm{VI}(f,S_1)$ and $\mathrm{VI}(f,S_2)$ by solving a series of system of nonlinear equations, as well as a series of linear variational inequality problems.

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